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# Near Hagedorn dynamics of NS fivebranes or A new universality class of coiled strings

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ABSTRACT: We analyze the thermodynamics of NS fivebranes as the temperature approaches the NS fivebranes' Hagedorn temperature, and conclude that the dynamics of "Little String Theory" is a new universality class of interacting strings. First we point out how to vary the temperature of the near extremal solution by taking into account $g_s$ corrections. The Hagedorn temperature is shown to be a limiting temperature for the theory. We then compare the thermodynamics to that of a toy model made of free strings and find basic discrepancies. This suggests a need for a new class of string interactions. We suggest that this new universality class is characterized by a strong attractive self-intersection interaction, which causes strings to be coiled. This model might also explain why "Little String Theories" exist in at most 5+1 dimensions.

KEYWORDS: Superstring Vacua, Black Holes.

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## Contents





## 1. Introduction

One of the more puzzling objects in string theory is the NS fivebrane. In particular the worldvolume theory, the so-called "Little String Theory" (LST), is believed to be some kind of non-gravitational string theory. These theories were introduced in [1, 2] (see also [3], for a review see [4]). In particular in [2] the decoupled theory on a cluster of NS fivebranes was defined by taking the limit

$$g_s \longrightarrow 0\,, \qquad M_s \text{ fixed} \tag{1.1}$$

of string theory in the presence of the cluster. This definition, however, is rather indirect in the sense that it does not provide a microscopic description of the theory. Rather, it is closely related to the gravitational holographic dual [5] (for a review see [6]) of the theory.

The gravitational side of the holographic duality for this case [7] is the throat region of the CHS background [8]. This background includes a linear dilaton direction, an $R^6$ component and a WZW model at a level set by the number of fivebranes $N$. This duality was used to analyze the observable content of the theory at the origin of its moduli space [7], and along its flat directions [9]. This was done both for the type-II NS fivebranes, the heterotic NS fivebranes [10], fivebranes at orbifolds [11] and lower-dimensional related configurations [12]. Five brane thermodynamics was discussed in [13].

A complementary way of exploring "little string theories" was introduced in [14, 15] and developed further in [16, 17]. In this approach a discrete light cone quantization [19] was suggested, along the lines of Matrix theory [20]. This DLCQ description is in terms of a 1+1 sigma model on the ADHM moduli space. Even though some aspects of this model are well understood (either from AdS/CFT or from a direct field theory analysis [17]), it is not yet clear how to directly obtain LST quantities from it and in particular how to obtain a covariant microscopic description (we suggest a way of going to the "long strings" picture for this sigma-model in section 6). For another suggestion, involving quasi-local field theories, see [18].

Despite the difficulties, it is worthwhile to explore these theories for a variety of reasons. The basic point is that these are stringy theories without gravity and without a tunable string coupling. This makes them a very intriguing object to study. In addition [1], they are important for a Matrix description of M-theory on $\mathbb{T}^5$. Moreover, in the context of holographic duality the "little string theory"/CHS background duality behaves differently from the AdS/CFT duality, and therefore might be a first step towards understanding holography in asymptotically flat spaces [7].

Another interesting aspect of these theories is an unusual UV/IR relation. When the theories are taken along the flat directions [9, 21], such that the Higgs vev is $M_w^2$, then in addition to the expected massless fields ($N$ tensor(vector) multiplets for type IIA(IIB)), one finds additional massless states. In fact one finds an entire stringy tower of massive states with string scale $M_s/\sqrt{N}$, even in the limit $M_w \to \infty$. This is very puzzling because naively one expects the theory in this limit to split into $N$ copies of the theory of a single NS fivebrane. Rather the low energy theory is still sensitive to details of the very high energy states. This is somewhat reminiscent of effects in other non-local quantum theories, i.e. the theories on non-commutative geometries [22]. The difference is that the extra states here are propagating particles in Minkowski space and the theory is Lorentz invariant.

More recently, these theories have appeared in the context of holographic duals of confining gauge theories [23]. Some of the confining vacua of $\mathcal{N} = 4$ deformed to $\mathcal{N} = 1$ pure glue are described holographically using NS fivebranes. This suggests a role for "little strings" in the confinement picture, or more generally as an approximate description, valid at some finite energy interval, of the IR region of field theories.

In this paper we discuss the behavior of LST at high energy densities. The holographic dual to this configuration is the near horizon limit of the near extremal fivebranes [24, 25], which includes the CGHS black hole [26, 27]. We are interested in the euclidean black hole which means that we are discussing the canonical ensemble. From this background it is easy to extract that the theory has a Hagedorn density of states and hence a Hagedorn temperature (for discussions see [28, 9, 29]). We are interested at the behavior of the theory as the temperature approaches the Hagedorn temperature from below.

The behavior at high energies is sometimes expected to resemble weakly coupled string theory [30, 28]. We attempt to interpret the thermodynamics in terms of a gas of weakly coupled strings. To this end we begin, in the next section, by reviewing a class of models of free strings and the resulting thermodynamics. This is meant to be compared to qualitative features, and not necessarily precise quantitative ones, of the LST thermodynamics. We discuss possible different behaviors near the Hagedorn temperature, the validity of the canonical ensemble and other features.

In sections 3 and 4 we discuss the thermodynamics of the CGHS black hole. Section 3 is a review of known results at string tree level and section 4 discusses one-loop corrections, which allow us to go off the Hagedorn temperature and study the partition function as we approach it. The results strongly suggest that the Hagedorn temperature is a limiting temperature for LST, rather than a phase transition as in weakly coupled critical string theory.

Section 5 is somewhat of an aside — we discuss how the large fluctuations of the canonical ensemble manifest themselves in the near extremal solution.

In section 6 we try to interpret our results in terms of the dynamics of an almost free string and show that there is some qualitative difference between the two systems. We interpret this as indicating a new universality class of interacting strings. We then suggest that in this universality class the strings have a strong self-attractive potential. This makes long strings want to shrink, which is a first step towards explaining the thermodynamics of the NS fivebranes. By a simple combinatoric random walk model we argue that this phase cannot occur for strings above 5+1 dimensions, which explains the maximal dimensionality of "little string theory" (although it does point to the fact that a similar modification for the theory of membranes might lead to an interacting theory even in higher-dimensional spacetimes). Our analysis is based both on space-time considerations and on DLCQ considerations, and points to a new way of analyzing "long strings" in the D1-D5 system.

We conclude by summarizing the main results of the paper.

As we were completing this project, we received a paper [49] with some overlap with sections 3 and 4 in our paper. The interpretation of the result is, however, quite different.

## 2. Ensembles of weakly coupled strings

Let us review some aspects of the thermodynamics of critical string theory, which are relevant for us (these would be similar to [31, 32]). Of course, the notion of a thermodynamic equilibrium in a theory with gravity is ill defined, but as explained in [31] it is justified in the weak string coupling limit, where some questions can still be addressed. The question we are interested in is that of the high energy density of states (at the sphere level). We do so with an eventual goal of examining what aspects of this high energy spectrum remain valid in "little string theory". Note that since LST does not contain gravity, a thermal equilibrium is well defined even though there is no weak coupling.

Since the discussion of qualitative features of free strings is sufficient for us, we restrict our attention to the simplest model on the worldsheet, i.e. free bosons and fermions. However, we allow for an arbitrary central charge $\hat{c}_{\text{eff}}$[1] (in light cone), and an unknown string tension $M^2_{\text{eff}}$. We allow this freedom since the immediate information we have about LST is the Hagedorn temperature which only determines the combination

$$\frac{\hat{c}_{\text{eff}}}{M^2_{\text{eff}}} = \frac{4N}{M_s^2}, \tag{2.1}$$

where $N$ is the number of NS fivebranes and $M_s$ is the spacetime string tension. This can be derived from the euclidean near extremal solution by measuring the periodicity of the time direction, which turns out to be the same for all values of the energy density. For example, the free string model of LST in [30] has strings of central charge 4 (4 bosonic coordinates in light cone and their supersymmetry partners) and a tension $M_s^2/N$. However, the Matrix model (with a single unit of null momentum) suggests a tension of $M_s$ and central charge $4N$. Hence we would like to keep the central charge and tension arbitrary (subject to the constraint (2.1)), and see which one better fits the "data".

There are several ensembles that one might use. One can use either the microcanonical or the canonical ensembles, and then one can use either a compactified or uncompatified space. It is at times stated that the canonical ensemble is unreliable in a theory with a Hagedorn density because there are large fluctuations in thermodynamic quantities. For example, the fluctuations of the average energy are:

$$\frac{\langle E^2\rangle - \langle E\rangle^2}{\langle E\rangle^2} \sim 1 \tag{2.2}$$

or much larger (as we remind the reader shortly). However, the CGHS black hole can be formulated just as well in euclidean signature as in lorentzian one, therefore one expects the holographic relation to hold just as well in the euclidean/canonical

[1]The reduced central charge is defined as usual, $\hat{c} = \frac{2}{3}c$. We will refer to $\hat{c}$ as the central charge in the following.

ensemble case. In fact we will show that the CGHS black hole predicts large fluctuations just below the Hagedorn temperature. Henceforth we restrict our attention to euclidean signature and to the canonical ensemble.

One can also work either in compact on non-compact space, i.e. LST either on $\mathbb{R}^{5,1}$ or $\mathbb{R} \times \mathbb{T}^5$ where $\mathbb{T}^5$ has some volume $V$. We compute the thermodynamics in both cases and examine their behavior as $T \to T_H$. In the case of compact space we will also work in the limit $V \to \infty$, however we will always take $T \to T_H$ first. Remember that these two limits do not commute in a free string theory. The limit in which $V \to \infty$ first for fixed $T$ converges to the uncompactified limit, whereas when $T \to T_H$ first , for fixed V, the result is different. One obtains a result which is independent of $V$ and different from the non-compact case.

We will incorporate an arbitrary central charge by having 4 bosons and fermions (in light cone) associated with target space coordinates of the LST, and $D = \hat{c}_{\text{eff}} - 4$ compact bosons and fermions, which model "internal" degrees of freedom, in the scenario of a large-central charge. The spacetime directions will be taken to be either non-compact or compact with a large volume. The compactification radia of the internal CFT will remain fixed as we vary the temperature or the space volume. To keep the Hagedorn temperature fixed, the effective string tension scales as $M_{\text{eff}}^2 = {M_s}^2/\hat{c}_{\text{eff}}$.

The mass formula for closed strings is the familiar:

$$M^2 = 2(n+n') + \vec{L}^2 + \vec{L'}^2\,, \qquad n' - n = \vec{L}\cdot\vec{L'}\,, \tag{2.3}$$

where we set the effective string tension to one for now, and recover it later. The vectors $\vec{L}, \vec{L'}$ are momenta and windings for the compact fields, rescaled to set the radia to one. The jacobian of this rescaling cancels in the expression for the free energy. The resulting free energy is (approximating the discrete sum by an integral):

$$\beta\, F = V_5 \int d^5p \int d\vec{L}\, d\vec{L'} \int dn\, \rho(n)\, \rho\left(n + \vec{L}\cdot\vec{L'}\right) e^{-\beta\sqrt{p^2+M^2}} \tag{2.4}$$

with :

$$\rho(n) = \frac{1}{2^{(D+4)}}\, \frac{(D+4)^{(D+5)/4}}{n^{(D+7)/4}}\, e^{\pi\sqrt{n(D+4)}}\,. \tag{2.5}$$

Any potential divergence in the free energy near the Hagedorn temperature comes from a saddle point where $n \to \infty$, so we can expand the integrand in that limit. This gives:

$$\begin{aligned}\beta F = V_5 \int d^5p \int dn\, \frac{1}{4^{(D+4)}}\, \frac{(D+4)^{(D+5)/2}}{n^{(D+7)/2}}\, e^{2\pi\sqrt{n(D+4)} - \beta\,\sqrt{p^2+4n}} \times \\ \times \int d\vec{L}\, d\vec{L'}\, e^{\frac{\pi\sqrt{D+4}}{2\sqrt{n}}\vec{L}\cdot\vec{L'}}\, e^{\frac{-\beta}{4\sqrt{n}}(\vec{L}+\vec{L'})^2}\,.\end{aligned} \tag{2.6}$$

One can now perform the integrals over $\vec{L}, \vec{L}'$. Since these integrals are not divergent as the temperature approaches the Hagedorn temperature, one can substitute $\beta = \beta_H = \pi\sqrt{D+4}$.

The expression for the free energy now becomes:

$$\beta F = V_5 \int d^5p \int dn \, \frac{1}{4^{(D+4)}} \frac{(D+4)^{(D+5)/2}}{n^{(D+7)/2}} \, e^{2\pi\sqrt{n(D+4)} - \beta\sqrt{p^2+4n}} \times \\ \times \frac{1}{2^D}\left(\frac{8\sqrt{n}}{\sqrt{D+4}}\right)^D \tag{2.7}$$

which simplifies to the following expression (omitting D independent numerical factors):

$$\beta F = V_5 \int d^5p \int dn \, (D+4)^{5/2} \, n^{-7/2} \, e^{2\pi\sqrt{n(D+4)}} \, e^{-\beta\sqrt{p^2+4n}} \, . \tag{2.8}$$

Now, define $m^2 = 4n$. We restore dimensions by using an effective string mass $M_{\text{eff}}$, and use $D+4 = \hat{c}_{\text{eff}}$. This gives, again up to numerical factors:

$$\beta F = V_5 \int p^4 \, dp \int \frac{M_{\text{eff}}^5 \, dm}{m^6} \, \hat{c}_{\text{eff}}^{5/2} \, e^{\pi\sqrt{\hat{c}_{\text{eff}}} \frac{m}{m_{\text{eff}}}} \, e^{-\beta\sqrt{p^2+m^2}} \, . \tag{2.9}$$

We find that the free energy is extensive in $V_5$ and finite as $\beta \to \beta_H$, and that these statements are independent of the string model used (the precise value does depend on $\hat{c}_{\text{eff}}$, which might eventually help determine $\hat{c}_{\text{eff}}$ for LST, if other obstacles, described in this paper, are overcome first). Even though we calculated this behavior for a free worldsheet we expect it to hold for any critical string at $g_s = 0$ string.

The same calculations above can be repeated to find the free energy in the case of a large compact spacetime directions. One finds the qualitative behavior:

$$\beta F \sim \log(\beta - \beta_H) \tag{2.10}$$

and independent of $V$. Again, this behavior is the same for the two toy models that we are using, and we expect it to hold for every $\sigma$-model.

As mentioned above, in the compact space case the behavior near the Hagedorn temperature is very different from the non-compact case, and in particular the free energy is no longer extensive. The difference between the non-compact and compact case arises due to the dynamics of long strings, which are allowed to wind on any compact directions. Below we compare these results with LST thermodynamics and find that even for a compact space the free energy is extensive. This is a first hint towards the dynamics of LST — it is in a phase which suppresses long strings.

One can now discuss other thermodynamics quantities, in the two cases. In the non-compact model the energy stays finite near the Hagedorn temperature [31]. In the compact case the energy diverges as one approaches the Hagedorn temperature:

$$E = \frac{1}{\beta - \beta_H} \, . \tag{2.11}$$

In addition energy fluctuations are large, and behave as

$$\frac{\langle E^2\rangle - \langle E\rangle^2}{\langle E\rangle^2} \sim 1\,. \tag{2.12}$$

The large fluctuations associated with the canonical ensemble can also be attributed to the behavior of the long strings. Near the Hagedorn temperature a finite fraction of the energy is stored in a single string. This makes it increasingly difficult to equilibrate the string gas as one approaches the Hagedorn temperature.

Next, we would like to compare this behavior to the near-Hagedorn behavior of LST. We do so by studying near extremal fivebrane solution, to which we turn now.

## 3. Near extremal NS fivebranes

We review here the gravity background holographically dual to LST at a finite temperature. The zero temperature duality was discussed in [7]. Roughly, as one approaches the boundary (UV) the solution asymptotes (exponentially fast) to the weakly coupled throat region of the CHS background [8]. At the interior (IR) there are several interesting regimes, described by eleven-dimensional supergravity, which however will not concern us here.

At a finite temperature the solution used in [7] has to be replaced by a near extremal version. Though the exact solution is not known for all temperatures, it is becomes very simple in the limit that the energy density becomes large, since then the entire solution resides in the throat region and can be described using string theory. The near extremal fivebrane solution was written down in [24, 25]. In terms of appropriately rescaled quantities (after decoupling) the solution is (using the string frame):

$$\begin{aligned} ds^2 &= dt^2\left(1-\frac{\mu}{u^2}\right) + dx_i^2 + \frac{N}{u^2}\left(du^2\left(1-\frac{\mu}{u^2}\right)^{-1} + u^2 d\Omega_3\right), \\ e^{2\phi} &= \frac{N}{u^2}, \end{aligned} \tag{3.1}$$

where $u$ is related to the radial coordinate away from the brane, and $\mu$ is the energy density above extremality. The index $i = 1, \ldots, 5$ corresponds to directions along the brane, and $d\Omega_3$ is the metric of the unit 3-sphere.

The near extremal solution can be brought into a more familiar form by a redefinition of coordinates, $u = \sqrt{\mu}\cosh r$ (we also rescale the coordinates along the brane):

$$ds^2 = N\Big[\, dt^2 \tanh^2(r) + dx_i^2 + dr^2 + d\Omega_3 \Big]\,, \qquad e^{2\phi} = \frac{N}{\mu \cosh^2(r)}\,. \tag{3.2}$$

We identify the background to be made out of the two-dimensional black hole [26, 27], a level $N$ supersymmetric WZW and an $R^5$ component.[2] We use the euclidean version of this black hole to describe the canonical ensemble. Accordingly, the euclidean time variable is taken to be compact, with periodicity $2\pi$ in the conventions of (3.2).

The geometry of the two-dimensional black hole is that of a semi-infinite cigar. The asymptotic circle closes at the tip, which is located at $r = 0$. The string loop expansion is controlled by the value of the dilaton at the tip of the cigar:

$$g_{s,\mathrm{tip}}^2 = \frac{N}{\mu}\,. \tag{3.3}$$

We see therefore that the string loop expansion is expansion in inverse energy density. This is the primary reason that string loop corrections in this background can change qualitative features of the thermodynamics.

It is worth noting that issues of decoupling are not relevant here [36, 39]. If the asymptotic string coupling is small enough, although not strictly taken to zero, there will still be a large range of energies in which the horizon of the black hole will be in the throat region and our analysis will apply.[3]

The tree level thermodynamics of the solution (3.2) is well known. The system has a fixed Hawking temperature, $T_H \sim M_s/\sqrt{N}$, regardless of the energy. We calculate the tree level free energy in appendix A and confirm that $F = 0$ for each value of the energy density $\mu$ (parameterized by the value of the dilaton at the tip, $\phi_0 = \phi(r = 0)$).

We find then that for fixed boundary conditions, there is a family of solutions to the euclidean equations of motion. These solutions have degenerate action. It would seem then that the parameter $\phi_0$ has to be summed over, and represents a flat direction in the path integral. Summing over $\phi_0$ would result in a divergence. However, in [37] it was argued that the mode changing the parameter $\phi_0$ is a non-normalizable mode, and should not be summed over in the path integral.

The next step is to study the system at finite energy, namely study string loop corrections.[4] This is done in the next section.

# 4. First contributions to the free energy

## 4.1 Scalings of correction terms

We discuss now the corrections to the leading order action, and their effect on various thermodynamic quantities. Higher derivative terms in the ten-dimensional action were discussed in [34, 35]. They include the famous $R^4$ term [34], and many other

[2]The extremal solution is a linear dilaton direction times a WZW times $\mathbb{R}^6$.

[3]The distance from the tip of the cigar to the asymptotic region is proportional to $\sqrt{N}$ $\log(g_{s,\mathrm{tip}}/g_{s,\mathrm{asymp}})$.

[4]Some aspects of the quantum corrections to the CGHS black hole were considered in [41].

terms, related by supersymmetry. In order to organize the corrections to the thermodynamics, we keep the number of fivebranes, $N$, large and fixed, and arrange the possible corrections in powers of $N$.

We estimate therefore the scaling with $N$ of a general higher derivative term, and then discuss the known terms in [34, 35]. The metric as written above, eq. (3.2), is proportional, in our notations, to $N$ (in the string frame). Both $H$ and $g_{\rm str}^2$ are also proportional to $N$. Therefore, in the Einstein frame one has:

$$g_{\cdot\cdot} \sim \mu^{1/4} N^{3/4}\,, \qquad H_{\cdots} \sim N\,, \qquad e^{\phi} \sim N^{1/2}\mu^{-1/2}\,, \tag{4.1}$$

where the dots indicate the position of spacetime indices. As a check of this scaling, all the terms in the leading order action

$$\sqrt{g}\left[R + H^2 + (\partial\phi)^2\right] \tag{4.2}$$

scale uniformly like $N^3\mu$.

We now can discuss the scaling of the correction terms. The most general putative correction term has the schematic form:

$$\sqrt{g}\, e^{b\phi} R^k_{\cdots\cdot}\, \partial_\cdot^{\,l} H_{\cdots}^{\ \ t}\, g^{\cdot\cdot p}\,, \tag{4.3}$$

where both $b$ and $p$ can be negative or positive, and $l, k, t$ are positive. We take negative $p$ to indicate metric with lower indices.

There are some constraints on the general term (4.3). First, the index structure of the above term demands:

$$4k + l + 3t = 2p\,. \tag{4.4}$$

This guarantees that the term in the action is a scalar under general coordinate transformations.

In addition, suppose we are interested in a particular power $s$ of the energy $\mu$. The string loop expansion is controlled by the value of the dilaton in the tip of the cigar. Therefore, the integer $s$ is related to the genus of the string diagram giving rise to the particular correction term ($s = 1 - g$, where $g$ is the genus). For the general term above the power of $\mu$ is:

$$4s = 5 - 2b + k - p\,. \tag{4.5}$$

We now can discuss the $N$ scaling of the general term (4.3). The power of $N$ of such a term is denoted by $A$, and is given by:

$$4A = 15 + 2b + 3k + 4t - 3p\,. \tag{4.6}$$

In order to simplify this expression for $A$ we can use the two relations above to write $A$ in terms of the positive quantities $k, l, t$, giving:

$$A = 5 - s - k - \frac{l+t}{2}\,. \tag{4.7}$$

For a given genus, the power of $N$ is determined by $k, l, t$ , and the two constraints above can be used to determine the needed $p, b$ to complete the correction term (4.3).

There are possible corrections to the action already at string tree level. These $\alpha'$ correction terms are all of $s = 1$ form. For example, the $R^4$ term has $k = 4$, and therefore scales like $\mu N^0$. In fact, one can show that all the other leading order tree level corrections scale like $\mu N^0$.

We are more interested in the leading order corrections in the energy, coming from one-loop terms in string theory. All such terms have $s = 0$. For example, the one-loop $R^4$ term has $k = 4$, and therefore scales as $\mu^0 N$. By a direct check, all other one-loop terms scale similarly. This includes terms coming from up to 8 point scattering.

To summarize, the leading corrections in one-loop are all of the same order, and scale like $\mu^0 N$. Compared to the leading order action they are suppressed by $\mu N^2$. We assume these corrections are non zero. In this case they represent the leading order correction to the thermodynamics.

It is of some interest to calculate the coefficient of the corrections. In addition to confirming that it is non zero, its sign has some significance in the thermodynamics. We elaborate on this point below.

### 4.2 Corrections to $E(T)$

In the presence of corrections to the leading order action, the solution (3.2) deforms slightly. The deformation is controlled by the size of the corrections $\eta = 1/N^2\mu$. We are interested in the deformed solution asymptotically in the $r$ direction, so we can observe a shift in the temperature. One has schematically:

$$I = I_0 + \eta\, I_1 + \cdots , \qquad (4.8)$$

where $I_0$ is the leading order action (A.1), and $I_1$ are the leading order corrections at one-loop, discussed above.

To keep track of the scaling of corrections to the thermodynamics, we follow an outline of the calculation. An exact calculation would require a detailed knowledge of the perturbed action $I_1$. Rather, we are interested in extracting the dependence of the temperature correction on the parameter $\eta$.

We are interested in the change in the asymptotic value of $G_{tt} = h^2(r)$. The equation of motion for this metric component reads:

$$h'' - 2\, h'\, \phi' = \eta\, \frac{\delta I_1}{\delta G_{tt}} \, . \qquad (4.9)$$

We define the right-hand side of this equation to be the source, denoted by $J_1$.

Far enough from the tip, the background becomes approximately the linear dilaton vacuum. We define the small fluctuations:

$$h = 1 + \delta h \, , \qquad \phi = -r + \log 2 + \phi_0 + \delta\phi \, . \qquad (4.10)$$

Here we neglect the mixing with other small fluctuations. A complete calculation would require, of course, diagonalizing the complete matrix of quadratic fluctuations in this background. Clearly the complete calculation retains the scaling with $\eta$ demonstrated here.

One gets then the following equation for the small fluctuations:

$$\delta h'' + 2\,\delta h' = J\,. \tag{4.11}$$

We note that the dilaton fluctuations drop out asymptotically, in the linear dilaton regime. The source in this equation $J = J_0 + \eta J_1$ consists of two terms. The first term $J_0$ is independent of the energy density, and is of no interest to us. The second term $J_1$ is the one-loop correction defined above.

The solution to equation (4.11) is a linear combination of the two solutions of the homogeneous equations. Therefore:

$$\delta h = a_1 + a_2\, e^{-2r}\,. \tag{4.12}$$

The two a-priori independent coefficients are determined by demanding regularity of the metric near the origin, as usual when determining the temperature. We are interested in the coefficient of the constant solution, $a_1$, which affects the asymptotic radius, and hence the temperature.

The coefficient $a_1$ is given by an overlap integral of the constant solution with the source $J$. The only dependence of $a_1$ on $N, \mu$ appears explicitly in the parameter $\eta$. Therefore $a_1$ can be written as a series expansion in the parameter $\eta$. The $\eta$ independent part only renormalizes the value of the Hagedorn temperature. The leading dependence on the energy density $\mu$ comes through:

$$\delta h(r) \sim \eta \qquad \text{as } r \longrightarrow \infty\,. \tag{4.13}$$

The shift in $h(r)$ gives the following correction to the inverse temperature.[5]

$$\frac{\beta - \beta_H}{\beta_H} \sim \eta\,. \tag{4.14}$$

The coefficient of the correction becomes significant at this point. If the proportionality constant in the last equation is positive, the system approaches the Hagedorn temperature from below, at high energies. Asymptotically the system has a positive specific heat. Therefore the canonical ensemble is well defined, though it has large fluctuations as explained above.

The scenario of a negative coefficient is very different: the system approaches the Hagedorn temperature from above, and has a negative specific heat. The canonical ensemble does not exist for high energies. We assume this is not the case and the coefficient is positive.

[5] There is also a correction to the relation between the parameter $\mu$ and the energy density, but it has a subleading effect in the string loop expansion.



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This gives the following modified relation between the temperature and the energy density:

$$\mu \sim \frac{N^{-5/2} {M_s}^5}{\beta - \beta_H}\,, \tag{4.15}$$

where ${M_s}^2$ is the spacetime string tension. Every string model will have to reproduce this expression, including the scaling with $N$.

### 4.3 Corrections to the partition function

The modified energy-temperature relation determines the thermodynamics of the system. Such a relation is expected to arise from an effective action of the form:

$$\beta\, F \sim \log(\beta - \beta_H) \sim \log(\mu)\,. \tag{4.16}$$

This kind of contribution comes from a dependence of the effective action $I = \beta F$ on the logarithm of $g_{\rm str}$, the string coupling at the tip. Since this dependence might seem unexpected, we demonstrate how it can be generated by the one-loop perturbation.

The leading perturbation to the effective action is a sum of two terms. The first comes from the substituting the deformed solution in the original action, $I_0$. The second term comes from the action $\eta I_1$, evaluated on the unperturbed solution. Logarithmic contribution can arise from terms in the integrand that are constant in the linear dilaton regime. Since the length of the linear dilaton regime is proportional the $\log(g_{\rm str})$, as shown above, this gives the desired effect.

Terms in the integrand which are constant in the linear dilaton regime are quite natural. We demonstrate here some possible sources for them. For example, the deformed solution for the metric component $G_{tt}$ is given by (4.12). This gives the following increment to the tree level effective action:

$$\delta I = \beta \int dr\, e^{2r} \left[-2\, \delta h'' + 4\, \delta h + \cdots\right]\,. \tag{4.17}$$

The integrand in this expression contains a constant term, where the decay of the fluctuation $\delta h$ is compensated for by the prefactor. There are also divergent terms, proportional to the temperature shift. This terms are regulated by considering differences in the action, as demonstrated in evaluating the leading term above. This procedure can also give rise to constant terms.

Another set of contributions to the free energy comes from a direct substitution of the original solution into the new action. These contributions can also contain constant terms in the integrand. To demonstrate this we need to consider the general correction term, evaluated in the linear dilaton background. In this background the only dependence on the radial coordinate is through the dilaton profile. Furthermore, derivatives of the dilaton are constant. The only dependence on the radial coordinate comes from the exponential of the dilaton. One-loop terms come with the dilaton



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exponential raised to zeroth power. We find therefore that the corrections terms, evaluated in the linear dilaton regime, are all constant, and give rise to the desired logarithmic dependence.

Therefore, logarithmic behavior of the free energy is natural in the linear dilaton throat. This logarithmic behavior of leads to large fluctuations in the energy. One gets, as was discussed above:

$$\frac{\langle E^2\rangle - \langle E\rangle^2}{\langle E\rangle^2} \sim 1\,. \tag{4.18}$$

The coefficient here is of order $N^0$. We elaborate on the issue of large fluctuations below.

In a critical string theory, the one-loop contribution results in effects that are quite different from the ones found above. At temperatures near the Hagedorn temperature there is a mode, winding around the euclidean time circle, which becomes light. The effective description of this mode is discussed in [31]. The situation is similar to the one described by Rohm [40], when considering Sherk-Schwarz supersymmetry breaking in string theory. The breaking by the boundary conditions on the circle results in a dilaton tadpole, and a one-loop cosmological constant. The modified equations of motion have no longer a static solution.

We now argue that the effects discussed in [31, 40] are subleading to the ones discussed above. The effects of SUSY breaking by the boundary conditions on the euclidean time circle can be computed in supergravity, as the system is at a low temperature compared to the string scale. The induced cosmological constant is of order $(\beta/l_p)^8$ compared to the leading order action. In the present background this gives a suppression by $N^3\mu$. To compare, the effects discussed above are suppressed by $N^2\mu$, as compared to the leading order action.

## 5. Another look at large fluctuations

The expression for the free energy, or the energy as a function of the temperature, reveals a problem in using the canonical ensemble when studying the thermodynamics of the system. The statistical fluctuations in the energy density are found to be:

$$(\Delta\mu)^2 \sim N^0\mu^2\,. \tag{5.1}$$

As one approaches the Hagedorn temperature from below, (normalized) fluctuations in the energy remain finite. This makes the canonical ensemble not very useful when calculating thermodynamic averages. Physically, this stems from the attempt to keep a system with an exponential density of states in an equilibrium with a heat bath.

However, the canonical ensemble is still holographically dual to the euclidean black hole, and it is useful to study these large fluctuations in the holographic description. At first sight, one might expect that the mode that changes the energy of the system becomes nearly massless at high energies, leading to the aforementioned fluctuations. However, as mentioned above, this mode is found to be a non-fluctuating mode [37]. We present here an alternative picture for the large uncertainty of the energy. A related picture has appeared before in [41].

The system has a large number of normalizable, fluctuating modes. Their specturm was written in [38], and the behavior in the linear dilaton regime was studied in [39]. The action for small fluctuations around a linear dilaton background is:

$$I = \int dr\, e^{2r} \left[G^{ij}\, \partial_i\psi\, \partial_j\psi\right] . \tag{5.2}$$

Here $G_{ij}$ is the string frame metric, and $\psi$ is a typical fluctuation, for example a fluctuation in the metric polarized along the brane directions. We study the radial profile of the fluctuations. Our radial coordinate is related to [39] as $z = \sqrt{N}r$. We also set the string tension to one for now. The modes $\psi$ were normalized to absorb any additional factor in the action.

The modes of the action are parameterized by $s = N\omega^2$. Their asymptotic behavior is:

$$\psi_\pm(\omega) = \exp\left(\beta_\pm(\omega)\, r - i\,\omega t\right) \tag{5.3}$$

with

$$\beta_\pm = -1 \pm \sqrt{1 - N\,\omega^2}\,. \tag{5.4}$$

For energies above the gap $\omega_0 = 1/\sqrt{N}$, the modes are normalizable. Their fluctuation scale is set by their action. We now proceed to estimate the fluctuations of a single normalizable mode.

Assume we are working with a finite cutoff, such that the length of the tube is finite and equals $L$. Imposing boundary conditions at $r = L$, and regularity conditions at the horizon, results in a discrete spectrum. Generically, this spectrum has spacing in energy $\omega$ in the order of $1/\sqrt{N}L$. This assumption is invalidated if the regularity at the horizon sets a complicated, $\omega$ dependent, relation between the two solutions $\phi_\pm$. We assume this is not the case for generic normalizable modes.

We assume therefore that the boundary conditions pick a solution of the schematic form:

$$\psi(\omega) = e^{-r}\sin\left(\sqrt{N}\,\omega r\right)\sin(\omega t)\,. \tag{5.5}$$

The exact form of the trigonometric functions is immaterial.

For large energy modes the action is:

$$I(\omega) = \int dr\, \omega^2 \sin^2\left(\sqrt{N}\,\omega r\right) \sim L\,\omega^2\,. \tag{5.6}$$

The last step comes from averaging a rapid fluctuation over the length $L$, which is much larger than the period. We find therefore that with a finite cutoff $L$, the typical fluctuations of the mode $\psi(\omega)$ are given by:

$$(\Delta\psi)^2 = \frac{1}{L\,\omega^2}\,. \tag{5.7}$$

Each mode $\psi(\omega)$ couples to the mode which moves the tip of the cigar by a certain form factor. This coupling is computed at tree level, and is therefore a general function of the form $G(s) = G(N\omega^2)$. The fluctuations of the location of the tip, resulting from the fluctuations in all normalizable modes, are estimated to be:

$$(\Delta a)^2 = L\sqrt{N}\int d\omega\, G\left(N\omega^2\right)\frac{1}{L\,\omega^2}\,, \tag{5.8}$$

where we denote the radial position of the tip of the cigar (as defined by the zero of the metric component $G_{tt}$) by $a$, and its variance by $(\Delta a)^2$. The normalization factor is needed when converting a discrete sum with spacing $1/L\sqrt{N}$ into an integral. We note that the dependence on the cutoff $L$ drops off, as it should.

The form factor $G(s)$ is unknown, but is expected to falloff at large frequencies, in order to yield a finite result. Impose a large frequency cutoff $\Lambda$, and rescale $k = \omega\sqrt{N}$. This gives:

$$(\Delta a)^2 = N\int^{\Lambda\sqrt{N}}\frac{dk}{k^2}\,G\left(k^2\right)\,. \tag{5.9}$$

Under the assumption of convergence of the integral, the $N$ dependence disappears from the integral. The fluctuations in the location of the tip of the cigar are of order $\sqrt{N}$, in proper distance. Therefore, in our notations, fluctuations in the coordinate $r$ are of order one.

This translated, via the exponential relation between the radial location of the tip and the energy density, to the following:

$$(\Delta\mu)^2 \sim N^0\mu^2\,. \tag{5.10}$$

We see therefore that under reasonable assumptions about the behavior of the normalizable modes, their collective effects result in large variance of the energy density. This fits with the expectations of the dual configuration, namely the canonical ensemble of LST.

## 6. The coiled phase of strings

### 6.1 The discrepancies

We would now like to compare more explicitly the free string model and the results obtained from the black hole. The free string results are either:

$$\begin{aligned} &T \longrightarrow T_H\,, \qquad V \text{ arbitrary}\,, \qquad \langle E\rangle = \frac{1}{\beta-\beta_H}\,, \\ &T \longrightarrow T_H\,, \qquad V = \infty\,, \qquad \frac{\langle E\rangle}{V} = \text{finite}\,. \end{aligned} \tag{6.1}$$

Again, these relations are expected to be robust in critical string theory. Both relations are expected to be generic in the $g_s = 0$ string.

On the other hand in "little string theory" the relation we obtained is:

$$T \longrightarrow T_H\,, \qquad V \text{ arbitrary}\,, \qquad \frac{\langle E\rangle}{V} = \frac{1}{\beta-\beta_H} \tag{6.2}$$

(and we have neglected factors of $N$, and dimensions are corrected using $M_s$). We clearly see that the free string model misses qualitative features of the thermodynamics. Next we would like to see whether there is some natural modification of the string dynamics that will go towards explaining the near-extremal thermodynamics.

Let us examine the non-compact partition function to see what modification can give us the correct result. Schematically the non-compact space free energy from section 2 is (after integrating over the momenta)

$$\beta F \sim V_5 \int \frac{dm}{m^{7/2}}\, e^{-(\beta-\beta_c)m}\,. \tag{6.3}$$

We would like to examine what modification to the partition function can change the power $m^{-7/2}$ into $m^{-1}$. Of course, there is no unique answer, but for later use we point out that a possible modification is changing the energy by a $\ln(m)$ term. More precisely the correct mass of a state is given by

$$m_{\text{correct}} = m - \frac{5}{2\,\beta_c}\,\ln(m)\,. \tag{6.4}$$

In this case, we will obtain the correct behavior for the partition function.

The general motivation will be following. The mass of the string is roughly its length. We are therefore interested in a string interaction that reduces the energy of the string by $\ln(l)$. We will see that there are such natural interactions, and we will focus on a self-intersection interaction which does that. Because the string now self-attracts it prefers to be coiled rather then large, solving the problem of dominance of long strings, and suggesting that we should consider a new phase made out of coiled strings.

### 6.2 General considerations

We have seen that it is difficult to understand the near Hagedorn, i.e. high string excitation, behavior of LST in terms of a familiar string theory. In this section we would like to propose a different picture which, although we cannot make precise, seems to remedy the situation. The upshot will be a new phase of strongly interacting strings with new qualitative features.[6]

[6] Another possibility is the existence of open strings in the system, this was suggested to us by O. Aharony. The thermodynamics of open string sectors was recently discussed in [48].

Let us begin by highlighting two main properties that such a solution should have:

- First of all it should suppress long strings: as we have seen above, when space is compactified on a large torus, the contribution of long winding strings makes the free energy non extensive. In order to obtain an extensive answer, we would like to suppress the contribution of these strings to $S(E)$.

  Actually, this aspect is closely related to the non-gravitational nature of the theory. As emphasized by Susskind [42], one generally expects that in gravitational theories, objects that naively contribute many states to the entropy become large, such that their contribution does not violate the Bekenstein bound (For example, the ground state of the string grows when taking into account more and more oscillators [43]). Hence it is natural in critical string theory for highly excited strings to tend to be very large.

  In our case, since there is no gravity and there are off-shell observables, we expect the string to prefer to be coiled rather then long at large excitation number. This observation is model-independent, and relies only general aspects of the relation between the number of degrees of freedom in a large volume and the size of highly excited objects.

- Secondly, the modification should have some reasonable space-time interpretation. Suppose we have a long string, or two long strings, then a reasonable space-time interpretation requires that when pieces of the strings are far away then the force between them will fall off at least as fast as the exchange of massless particles.

The conclusion is that we would like to modify the string models that we used before, which were some CFT on a worldsheet, by a strong attractive interaction. The new string is such that it cannot be written in terms of a local worldsheet action (otherwise, we do not expect to evade the problems outlined above).

As an extreme, we can try and model the string with purely local interactions in space-time. In this case it would be a self-intersection interaction. We will argue that this is indeed what happens in the light cone frame. There are various kinds of self-intersection interactions that one can write down, but one expects that the most relevant attractive self-intersection interaction is simply such that one looses a fixed amount of energy for every self-intersection (in some regulated version of the worldsheet).

This is, of course, not precise for the LST. For example, two segments of strings that intersect at a very small angle such that they are almost parallel are almost BPS and hence there is no force between them, whereas anti-parallel strings attract each other. Hence the interaction is not as simple as we suggest. Note, however, that the average over these configurations certainly gives an average attractive interaction.



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Two more comments are due. First, we have emphasized the effects of self-intersections but clearly if there is a strong-intersection interaction, then each string in the thermal state interacts strongly with the background. How can we take into account this interaction? We do not have the complete answer to this question, but we can suggest the following observation. One expects that if one uses a "mean field approximation" in which one computes the single-string state statistics in a homogeneous background (encoded in the values of some order parameters), then the contribution to the energy of the string will be proportional to the length of the string. This means that the "mean field approximation" parameters of the background renormalize the string tension, but may not correct other terms. In particular the self-intersection term (which we will show gives the logarithmic correction in the exponent) is left uncorrected. The string tension may indeed be renormalized, but the Hagedorn temperature already measures the physical tension, after this renormalization has been taken into account (actually, a self-intersection interaction also renormalizes the string tension, and the same argument applies to this renormalization as well).

Finally, in our discussion of the interaction we have used some kind of intuition about locality. This might be dangerous in a theory with T-duality in which space-time is not a well defined object — do we mean the initial, say, torus or its T-dual? In our discussion what we mean is that in the large volume limit, there is an interaction with the properties discussed above, and in particular with approximate locality in the large space-time. There are many other non-local interactions, coming say from winding strings, but they decay as $V \to \infty$. This is true in critical string theory, and when we turn our attention to motivations Matrix description next, we will see that the behavior is very similar in LST.

### 6.3 Matrix model considerations

It is instructive to check whether this picture is consistent with the DLCQ description of LST [14, 15]. We are motivated by the fact that counting the single-string degeneracy is simplest in light cone, and hence we would like to use Matrix theory to analyze it.

The Matrix model for LST on $\mathbb{R}^{5,1}$ is the D1-D5 system which is a $(1+1)$-dimensional sigma model on the ADHM moduli space. The latter is parameterized by two integers: $N_0$ which is the number of instantons and $N$ which is the rank of the $\mathrm{U}(N)$ gauge group. In the Matrix interpretation $N_0$ measures the null momentum and $N$ the number of NS fivebranes. The Matrix model for the LST on $\mathbb{R}^{1,1} \times \mathbb{T}^4$ is the same D1-D5 system, where the D5 is now compactified on $\mathbb{T}^4$, and we will focus on this model in our discussion. This model is a sigma model on a target space which is a deformation of ${\mathbb{T}^4}^{N_0 N}/S_{N_0 N}$ where $S_l$ stands for the group of permutations of $l$ elements. The model is a deformation in the sense of not being at the solvable orbifold point. Rather, it is at a singular point in the moduli space of CFT. This point is the analogue of the $\theta = 0$ point of $\mathbb{R}^4/\mathbb{Z}_2$ [44]. Because of these singularities it is not clear how to analyze the model, but fortunately these singularities are rather mild as far as the current question is concerned. The reader is however warned that the analysis we present is rather speculative, but at this point we would like primarily to check consistency with the picture put forward above. The analysis also implies a new way of analyzing the D1-D5 system, and it will be interesting to explore it more rigorously further.

We would like to analyze the spectrum relevant in the Matrix framework (i.e. the correct energy scaling) and see whether it fits the line of thought explained in the previous subsection. If the model was the symmetric product then the analysis would have been straightforward. We could go to a picture of "long strings" by going to the twisted sector of the $S_{N_0}$ part of the symmetric group and obtain a string with the correct scaling of energy, tension $M_s$ and central charge $4N$ (or we could go to even longer strings with by using the remaining $S_N$ symmetry to a string with central charge 4, but we will not need this stage here). This would give us a Hagedorn density, but we saw before that it does not reproduce the thermodynamics correctly. This is not surprising since the CFT is not at the orbifold point.

Nevertheless, we would like to argue that such long strings are a good starting point. The reason for that is that the entropy of strings far away from the singularity is much larger then the entropy of strings at the singularity. The reason is that the total central charge of the CFT is $4N_0N$, whereas the effective strings at the singularity have a much smaller central charge. This can be extracted from [17] in which the states at the singularities were described in terms of a 1+1 field theory written in terms of a $U(N_0)$ vector multiplet. The states at the singularity correspond to excitations along the flat directions of this non-abelian gauge theory, and hence their central charge is much smaller.

Hence, most of the states can be thought of as bulk states, i.e. these states for the most part are traveling away from the singularities of the sigma-model and are unaware of the singularity. In this sector one can try and go to long strings first and then consider the effect of the interaction. These long strings are the strings in spacetime and the interaction is precisely an interaction which is localized when the string intersects itself — i.e. it is an interaction of the same type that we were advocating above.

Hence the Matrix theory description lends support to the proposal that the modification is a strong self-intersection interaction. This analysis also suggests a new way of analyzing the D1-D5 system.

### 6.4 A random walk model

Let us now try and estimate how a self-intersection interaction in light cone can change the partition function of a string theory. We will not do so precisely but rather point to a relation between this model and a certain model of self-attractive random walks. We will perform most of the computations in the latter model.

Modelling dynamics by a random walk is familiar from polymer physics [45]. In the context of string theory Horowitz and Polchinski [46] analyzed corrections to the size of an excited string state using such methods in the context of the black-hole/excited string correspondence principle.[7] Although most of the analysis there is not directly relevant to our case,[8] it is worth noting that they also find significant effects which lead to a contraction of string states.

Also, we will not be working precisely in the DLCQ of LST but rather in a simplified model in which there are 4 transverse (to the lightcone) coordinates and a simple self-intersection interaction term, i.e. the energy decreases by some fixed amount when the string self-intersects in these coordinates. Other then this interaction term, one usually takes the action for the string to be its area. We will take a simplified model in which the weight of the string in the partition function is its length. This is so because we are counting physical states in lightcone, and we expect some equipartition between the kinetic and potential term. Hence there would still be a linear relation between the average size of the string as it fluctuates in some quantum state and the energy of that state (after all in lightcone the theory is a collection of oscillators). In any case, the skeptic reader can view the following analysis as a toy model which on the one hand reproduces some aspects of the partition function, and on the other hand is convenient for the analysis of the self-intersection interaction.

Under these assumption the partition function, without self intersections, at some inverse temperature $\beta$ will be

$$\int dl\, f(l)\, e^{\beta_c l-\beta l}\,, \tag{6.5}$$

where $l$ is the length of the string ($f(l)$ is a function which determined the degeneracy for a given length $l$). In relation to the usual quantization of strings we see that in order to match to the usual free oscillator picture we need to assume that at leading order $l\sim\sqrt{n}$.

We would like to regulate this partition function in a way that captures its interpretation as strings in space-time. The way to do so is to approximate it as a sum over random walks, which will also enable us to more precisely define what we mean by the number of self-intersections. The length $l$ will now become a discrete variable and the partition function is a sum over all random walks with $l$ steps with weights $e^{-\beta l}$. We will also assume the simplest form of a random walk, i.e. a cubic lattice with nearest neighbors jumps with equal probabilities.

[7]There is, however, a difference in that there if the string moves in d+1 dimensions, then the random walk is in d dimensions. In our case we are motivated by the lightcone picture to use a local intersection interaction for the 4 transverse coordinates in light cone.

[8]An important part of interaction there is a long-range gravitational self attraction, which we do not have here. Also, it is not at all clear what might be the interpretation of the thermal scalar in our context.



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Adding now the attractive interaction, the partition function becomes

$$\sum_{\text{random walks}} e^{-\beta(l-g_{\text{rw}}J)}\,, \tag{6.6}$$

where $J$ is the number of self-intersection

$$J=\sum_{i,j=1,\ldots,l}\delta_{w(i),w(j)} \tag{6.7}$$

and $w(i)$ denotes the location of the random walk at time $i$. $g_{\text{rw}}$ is some definite number which is determined by micro-physics and therefore we cannot estimate.

We come now to the main purpose of the random walk model. Evaluating the number of self intersections for a simple random walk of length $l$ and regarding it as a correction to the energy of the random walk, we will obtain a logarithmic correction to the energy. Of course, without knowing the coefficient $g_{\text{rw}}$ above we will not be able check whether the logarithm in the exponent gives eventually the right power of $l$ needed to correct the degeneracy of the states, but it is interesting to get a logarithmic correction at all. The reason is that in any higher dimensions there are no logarithmic terms in the self-intersection number. Hence 4 coordinates in lightcone is the maximum number for which one would expect to see this type of universality class of strings. Fortunately this is precisely the maximal dimension of LST.

Finally let us explain how the logarithm comes about. We would like to evaluate $\langle J\rangle_l$ on a closed loop of length $l$. We will use a Feynman diagram like technique (combinatoric computations as well as diagramatic techniques are described in [47]. An explicit formula for self-intersection number, for open random walks in various dimensions, appears in Brydges and Slade [47]). Since we are evaluating a single insertion of $J$, the dominant contributions are random walks in which start at point 0, propagate to some point X after $i$ steps, return to that point after additional $j$ steps and then return to the origin after additional $l-i-j$ steps. This diagrams contributes (The factor $l^2$ in front is due to normalizing by the probability that the random walk will return to the initial point — i.e. will be closed):

$$\langle J\rangle_l \propto l^2\int_{i+j<l} di\,dj\,dx\,e^{-x_1^2/i}\,\frac{1}{i^2}\,\frac{1}{j^2}\,e^{-x^2/(l-i-j)}\,\frac{1}{(l-j-i)^2}\,, \tag{6.8}$$

where we have made the discrete sum over steps into an integral but require $i,j>1$ (we are not careful with numerical coefficients). It is straightforward to evaluate this integral and which yields in the large-$l$ limit:

$$\langle J\rangle_l \propto l+2\ln(l)+O(1)\,. \tag{6.9}$$

The first term renormalized the string tension, and the second term is precisely what we wanted above — a change in the energy of the string that is proportional to the logarithm of its space-time length.

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## 7. Conclusions

In this paper we discussed the thermodynamics of "little string theory" at high energies. Studying string loop corrections to the holographic dual we were able to go slightly off the Hagedorn temperature and probe the physics as we approach that critical temperature.

The main surprise is that this physics is inconsistent with a picture of free closed strings at high energies. The discrepancies are fundamental and generic to all free ($g_s = 0$) string models. This provides a possible clue to the dynamics of the "little strings" at high energies.

The main discrepancy has to do with extensivity of the thermodynamics when the system is put at a finite volume. Free strings do not give extensive results, due to the dominance of long strings at high energies. The holographic duality teaches us that the "little strings" do give an extensive result. This is a first hint towards understanding dynamics in "little string theory": the "little strings" do not form a single-long string at high energies, rather they prefer to clump in small coils.

We argue for a simple model which reproduces this behavior. This model involves strings (in the lightcone) which interact only when they self-intersect. This suggestion is motivated by several observations. First, it seems that the interactions of the spacetime strings as seen in the DLCQ description yields an effective interaction of this form. Second, it is consistent with a local (on large scale) dynamics in spacetime. In addition, the interaction is different enough from the free string picture, so that a different qualitative behavior is possible.

Indeed, a simple analogous random walk model shows significant changes when an attractive self-intersection interaction is added. The changes look qualitatively similar to what we need: strings tend to become "coiled", and the partition function gets corrections similar to the ones needed to reproduce the thermodynamics of "little strings theory".

Clearly, more work is needed to establish this claim. A direct relation to a random walk model is needed if one is to reproduce exact, quantitative features of "little string theory". Perhaps working in the DLCQ description, this relation can be clarified. We hope to return to these issues in the near future.

## A. Vanishing of the leading order free energy

We calculate here the leading order thermodynamic quantities. We use the canonical ensemble, which corresponds to a euclidean black hole configuration with a compact time direction. The basic quantity to calculate is the free energy, which is obtained from the value of the action on-shell [33].



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We start with the ten-dimensional Einstein frame action:

$$I = \frac{1}{16\,\pi\,G_{10}}\left[\int_M d^{10}x\,\sqrt{g}\left(R - \frac{1}{2}\,(\partial_\mu\phi)^2 - \frac{1}{12}\,e^{-\phi}H^2\right) + 2\int_{\partial M} K\right]. \qquad \text{(A.1)}$$

To fix the ambiguity in the total action we need a prescription for boundary terms in (A.1). Following [33], we choose the total action such that fixing the values of the metric at infinity, but not its normal derivatives, is allowed. The standard boundary term, written above, involves the trace of the second-fundamental form of the boundary.

We now transform to string frame metric $G$ by a conformal transformation, $g_{..} = e^{-\phi/2}G_{..}$, where $H_{...}$ remains unchanged in the transformation. Also, to compare to notations in [26], we write $\Phi = -2\phi$.

In addition to the standard string frame bulk action, there is an additional boundary term, which is, for the sphere at infinity:

$$-\left(\frac{9}{4}\right)\sqrt{G}\,e^{\Phi}\,G^{rr}\,\partial_r\Phi\,, \qquad \text{(A.2)}$$

where $r$ is the radial direction.

Now we perform dimensional reduction. Denote by $V_5$ the volume of the non-compact 5 dimensions, and by $V_{\rm sph}$ the volume of the 3-sphere. We choose the following ansatz for the fields:

$$ds^2 = \left(ds_2{}^2 + dx_i{}^2 + N d\Omega_3\right), \qquad \Phi = \Phi(r)\,, \qquad H = N\,dV\,. \qquad \text{(A.3)}$$

Here $i = 1, \ldots, 5$ are flat directions along the brane, and $d\Omega_3$ denotes the standard metric on $S^3$. $dV$ is the volume element on the 3-sphere. This ansatz covers both the extremal and non-extremal fivebrane solution, therefore it is suitable when calculating the effective action.

One then gets

$$I = \frac{V_5\,V_{\rm sph}}{16\,\pi\,l_s{}^8}\int d^2x\,\sqrt{G}\,e^{\Phi}\left[R + G^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi) + \frac{2}{N}\right]. \qquad \text{(A.4)}$$

The prefactor is normalized in [26] to be 1, by a shift of the dilaton $\Phi$. We set the prefactor to 1, and recover it later.

We work in the ansatz for the two-dimensional metric:

$$ds_2{}^2 = N\left(dr^2 + h^2(r)\,dt^2\right). \qquad \text{(A.5)}$$

Here $t$ is euclidean time, compactified with a period $\beta l_s$, where $\beta$ is dimensionless. The physical temperature is then $1/\sqrt{N}\beta l_s$.

The action should include also boundary terms discussed above. The boundary terms in the ten-dimensional string action, written in the present ansatz, give the following terms:

$$I_1 + I_2 = \frac{2}{N}\,\beta\,e^{\Phi}\,h'\,. \qquad \text{(A.6)}$$

Both are to be evaluated at the boundary at infinity only, where the Dirichlet boundary conditions have to be imposed.

The bulk two-dimensional action (A.4) depends on second derivatives of the function $h$. In order to find the equations of motion we integrate by parts. This results in a boundary term we denote $I_3$:

$$I_3 = \beta \sqrt{G}\, e^{\Phi} \left[ G^{\mu\nu}\, \Gamma^r_{\mu\nu} - \Gamma^\mu_{r\mu} \right] . \tag{A.7}$$

The bulk action is then:

$$I = \beta \int dr\, e^{\Phi} \left[ 2h'\, \Phi' + 4h + h\, \Phi'^2 \right] , \tag{A.8}$$

where prime denoted derivative with respect to $r$.

The value of the action on shell is also a total derivative. This is easily shown using the equations of motion of the field $\Phi$. Therefore the two-dimensional bulk action (A.4), when evaluated on shell, equals the following terms, evaluated at the boundary:

$$I_3 + I_4 = \frac{1}{N}\, \beta\, e^{\Phi}\, h\, \Phi' . \tag{A.9}$$

The boundary here includes the black hole horizon as well, due to the different origin of this boundary term. This is still to be regarded as a bulk action, though for this particular solution it reduces to boundary terms.

It is easy to show that the following solves the equations of motion:

$$h(r) = \tanh(r) , \qquad \Phi(r) = 2 \log\cosh(r) + a . \tag{A.10}$$

Where $a$ is an arbitrary constant. This is the near extremal solution (3.2). The action of this solution should be compared with the vacuum configuration:

$$h(r) = 1 , \qquad \Phi = 2\, r + a - 2 \log 2 . \tag{A.11}$$

The additive constant in $r$ is chosen so that the field $\Phi$ has the same asymptotic behavior.

The effective action of the black hole configuration is found by a direct substitution in the above expressions. On general grounds one gets:

$$I_{\text{total}} = \beta F = \beta E - S . \tag{A.12}$$

Here $F$ is the free energy of the configuration, $E$ is the average energy and $S$ is the entropy. The first term $\beta E$ comes from boundary contribution, and the second one from a bulk contribution.

The bulk contribution comes from eq. (A.9). This leads to a divergent result for the black hole (A.10), but has to be compared to the action of the linear dilaton vacuum. This gives:

$$I_3 + I_4 = -\frac{1}{N}\, \frac{V_5\, V_{\text{sph}}}{8\, \pi\, l_s^{\,8}}\, \beta\, e^{a} , \tag{A.13}$$

where we have restored the prefactor set to 1 above.

The terms that get a contribution from the boundary at infinity are given in (A.6). They give the action:

$$I_1 + I_2 = \frac{1}{N} \frac{V_5\, V_{\rm sph}}{8\,\pi\, {l_s}^8} \beta e^a\,. \tag{A.14}$$

Combining the above gives $F = 0$ for every $a$. Also one can read off the energy density and the entropy:

$$\mu = \frac{E}{V_5} \sim \frac{N}{g^2} \frac{1}{{l_s}^6}\,, \qquad S = \beta_H\, E \quad \text{with } \beta_H = l_s \sqrt{N}\,. \tag{A.15}$$

This gives the expected Hagedorn behavior, and the correct relation between the energy density and the string coupling, as obtained by other methods.

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